An Optimal FIR Filter for Linear TIE Models of Local Clocks

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***Abstract*— In this paper, we present an optimal finite impulse response (FIR) filter for linear TIE models of local clocks. A comparison with the unbiased FIR filter is provided. Estimations are carried out for a local crystal clock using GPS-based sawtooth measurements. As a main conclusion, we notice that an optimal solution does not contribute too much to estimation accuracy and the unbiased FIR filter is definitely the best choice for estimation of the TIEs of local clocks.**

I. Introduction

Accurate and precise evaluation of the time interval error (TIE) of a local clock is a key problem in timekeeping. It is seemingly obvious from the standpoint of control that the more accuracy is obtained in TIE measurements and estimates, the lesser errors will occur while time dissemination [1] and clock synchronization [2].

The one pulse per second (1PPS) output of the Global Positioning System (GPS) receiver is commonly used as a time reference. The problem, however, arises of noise induced owing to the principle of the 1PPS output formation, temporary GPS time uncertainty, and signal propagation. It is known that noise in the 1PPS has a sawtooth structure, is uniformly distributed, and thus is non-Gaussian. Short-time noise in GPS-based sawtooth-less measurements is near Gaussian. However, temporary GPS time uncertainty has a nonstationary structure complicated with excursions by different satellites in a view. For the above reasons, the Kalman filter designed for white Gaussian noise may become unstable and produce estimates biased and noisy.

To avoid problems with Kalman estimates, an unbiased finite impulse response (FIR) filter was designed in [3] for linear TIE models typically associated with precise crystal and rubidium clocks. Note that, contrary to the infinite impulse response (IIR) structures such as the Kalman filter, FIR structures have inherent properties, such as bounded input/bounded output (BIBO) stability and robustness against temporary model uncertainties and round-off errors. The filter [3] was examined in [4] for applications in GALILEO and it was experimentally shown (Fig. 8 in [4]) that the mean square error (MSE) and Allan deviation in its estimates are both minimum among other filters examined. Moreover, minimum errors occur at the shortest average time. Because the TIE model is not always linear, the unbiased FIR filter was generalized in [5] to the K-degree TIE polynomial model and investigated in [6].

In this paper, we derive an optimal FIR filter for linear TIE models of local clocks and compare optimal and unbiased estimates. As will be clear in the sequel, the optimal solution does not contribute too much to estimation accuracy and unbiased FIR filters seem to be the best choice for GPS-based timekeeping.

II. Mean Square Error of FIR Estimates

Assume that the local clock TIE x_n, where n representes discrete time points following with the time step (sample time) τ, is measured using the GPS timing receiver in the presence of additive and zero-mean noise v_n induced by the receiver, temporary GPS time uncertainty, and signal propagation. The measurement can thus be written as

$$\xi_n = x_n + v_n\,. \tag{1}$$

Provided the estimate $\hat{x}_n$ of the TIE x_n, the estimate error may be evaluated by

$$\varepsilon_n = x_n - \hat{x}_n \tag{2}$$

and the MSE associated with (2) by

$$E\{\varepsilon_n^2\} = E\{[x_n - \hat{x}_n]^2\}\,. \tag{3}$$

If the TIE model is linear for the filter memory (horizon) of N points, it projects ahead from $n-N+1$ to n as

$$x_n = x_0 + y_0 \tau n\,, \tag{4}$$

where $x_0 \equiv x_{n-N+1}$ is an initial time error and $y_0 \equiv y_{n-N+1}$ is an initial fractional frequency offset.

Without losing generality, one can neglect x_0 and provide the real time FIR estimate $\hat{x}_n$ of x_n by the discrete convolution operator $\mathcal{C}$ as

$$\begin{aligned}\hat{x}_n &= \mathcal{C}\xi_n = \mathcal{C}(y_0\tau n + v_n)\\ &= y_0\tau\sum_{i=0}^{N-1} h(i)(n-i) + \sum_{i=0}^{N-1} h(i) v_{n-i}\,,\end{aligned} \tag{5}$$

where the impulse response $h(i)$ obeys an inherent property

$$\sum_{i=0}^{N-1} h(i) = 1\,. \tag{6}$$

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By (5) and zero-mean noise v_n, the MSE (3) becomes

$$\begin{aligned}E\{\varepsilon_n^2\} = E\Bigg\{\Bigg[y_0\tau n - y_0\tau\sum_{i=0}^{N-1}h(i)(n-i)\\ -\sum_{i=0}^{N-1}h(i)v_{n-i}\Bigg]^2\Bigg\}\\ = y_0^2\tau^2\left[n-\sum_{i=0}^{N-1}h(i)(n-i)\right]^2\\ +\sum_{i=0}^{N-1}\sum_{j=0}^{N-1}h(i)h(j)E\{v_{n-i}v_{n-j}\}\\ = y_0^2\tau^2(n-\mathbf{n}^T\mathbf{W})^2+\mathbf{W}^T\mathbf{R}\mathbf{W}\,,\end{aligned} \quad (7)$$

where

$$\mathbf{n} = \left[\begin{array}{cccc} n & n-1 & \dots & n-N+1 \end{array}\right]^T, \quad (8)$$

the FIR matrix is

$$\mathbf{W} = \left[\begin{array}{cccc} h(0) & h(1) & \dots & h(N-1) \end{array}\right]^T, \quad (9)$$

and $\mathbf{R}$ is the $N \times N$ covariance matrix of v_n.

Owing to slowly changing TIE of the clock, a large horizon $N \gg 1$ is commonly used. By virtue of that, noise induced by the receiver may be considered to be delta-correlated [5]. This degenerates $\mathbf{R}$ to the diagonal form with the components $R_{i,i} = \sigma_v^2 = E\{v_i^2\}$ and we have

$$E\{\varepsilon_n^2\} = y_0^2\tau^2(n-\mathbf{n}^T\mathbf{W})^2 + \sigma_v^2\mathbf{W}^T\mathbf{W}\,. \quad (10)$$

Let us analyze either (7) or (10). It follows that the MSE is composed with two components. The first term represents the square time shift (bias) in the estimate, bias_n^2, and the second one the variance σ_n^2. In view of that, we rewrite (10) as

$$E\{\varepsilon_n^2\} = \mathrm{bias}_n^2 + \sigma_n^2 \quad (11)$$

and notice that, by the physical reasons, the components in (11) are uncorrelated and independent.

Fig. 1 demonstrates what happens with the MSE if we use different estimators. It is known that simple averaging associated with the uniform FIR

$$h_0(i) = \begin{cases} \frac{1}{N}, & 0 \le i \le N-1 \\ 0, & \text{otherwise} \end{cases} \quad (12)$$

produces minimum possible noise among all other estimators, including the Kalman filter. Therefore, simple averaging is optimal in the sense of the minimum produced noise. However, bias, by (12), is 50% for linear models. The (left) vector diagram in Fig. 1 illustrates this estimate.

Contrary to simple averaging, an unbiased FIR estimate is obtained by [3], [5]

$$h_1(i) = \begin{cases} \frac{2(2N-1)-6i}{N(N+1)}, & 0 \le i \le N-1 \\ 0, & \text{otherwise} \end{cases} \quad (13)$$

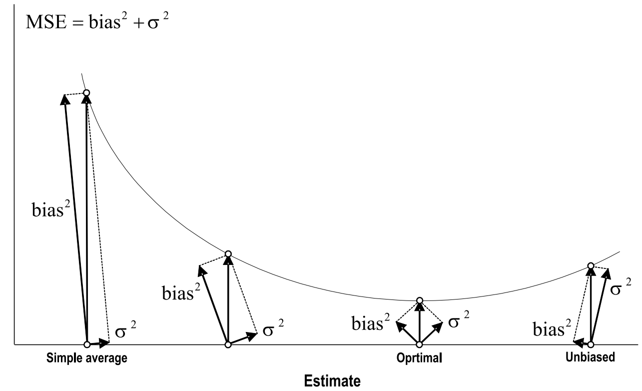

Fig. 1. MSE, square bias, and variance in different kinds of estimates.

with minimum bias (theoretically zero) and a greater amount of noise. The (right) vector diagram in Fig. 1 illustrates this case.

Clear that there must be an optimal estimate placed between simple averaging and unbiased estimates allowing for the minimum MSE. An example is the Kalman filter that, however, cannot degenerate to simple averaging when the TIE model becomes uniform. The latter is owing to the IIR structure and a limitation in applications to precise clocks. We show below that this disadvantage is not peculated to FIR filters that, under the model and measurements conditions, can produce estimates optimal at any point in Fig. 1 between simple averaging and unbiased estimates.

III. An Optimal FIR Filter

To design an optimal FIR filter, the MSE (10) must be minimized to produce conditions of optimality. Because the TIE model is still linear, minimization may be provided by adjustment of the linear FIR (13) aimed at decreasing noise at the expense of bias.

A. A Modified Impulse Response

Let us represent (13) on a horizon of N points as

$$h_1(i,\eta) = \begin{cases} \frac{1}{N}\left[1+\eta\frac{3(N-1)-6i}{N+1}\right], & 0 \le i \le N-1 \\ 0, & \text{otherwise} \end{cases}, \quad (14)$$

where η is a coefficient such that, by $\eta = 0$, (14) degenerates to simple averaging (12) and, by $\eta = 1$, (14) becomes unbiased (13). The following theorem verifies that (14) is still the impulse response.

Theorem 1: Given an arbitrary real $-\infty \le \eta \le \infty$. Then (14) is the impulse response satisfying (6).

Proof: Test (14) by (6),

$$A = \sum_{i=0}^{N-1}\frac{1}{N}\left[1+\eta\frac{3(N-1)-6i}{N+1}\right]$$

$$= \frac{1}{N}\sum_{i=0}^{N-1} 1 + \eta \sum_{i=0}^{N-1} \frac{3(N-1)-6i}{N+1}$$
$$= 1 + \eta \times 0 = 1 \,.$$

For any η, (14) satisfies (6) and the proof is complete. ■

B. Condition of Optimality

Now, rewrite (10) for the η-dependent matrix (9) as

$$E\{\varepsilon_n^2\} = y_0^2\tau^2[n - \mathbf{n}^T\mathbf{W}(\eta)]^2 + \sigma_v^2\mathbf{W}^T(\eta)\mathbf{W}(\eta)\,. \quad (15)$$

The FIR filter would be optimum in the sense of the minimum MSE if (15) is minimized by η. To find a condition of optimality, we equate to zero the derivative of $E\{\varepsilon_n^2\}$ taken with respect to η and provide the transformations as in the following,

$$\frac{\partial}{\partial\eta}E\{\varepsilon_n^2\} = 0$$
$$= \frac{\partial}{\partial\eta}\{y_0^2\tau^2[n - \mathbf{n}^T\mathbf{W}(\eta)]^2 + \sigma_v^2\mathbf{W}^T(\eta)\mathbf{W}(\eta)\}$$
$$= -2y_0^2\tau^2[n - \mathbf{n}^T\mathbf{W}(\eta)]\frac{\partial}{\partial\eta}[\mathbf{n}^T\mathbf{W}(\eta)]$$
$$+\sigma_v^2\frac{\partial}{\partial\eta}[\mathbf{W}^T(\eta)\mathbf{W}(\eta)]\,. \quad (16)$$

By (14), a subfunction $\mathbf{n}^T\mathbf{W}(\eta)$ in (16) becomes

$$\mathbf{n}^T\mathbf{W}(\eta) = \sum_{i=0}^{N-1} h_1(i,\eta)(n-i)$$
$$= \sum_{i=0}^{N-1}\left[\frac{1}{N} + \eta\frac{3(N-1)-6i}{N(N+1)}\right](n-i)$$
$$= n - \frac{N-1}{2}(1-\eta)\,. \quad (17)$$

The derivative of (17) with respect to η yields

$$\frac{\partial}{\partial\eta}[\mathbf{n}^T\mathbf{W}(\eta)] = \frac{N-1}{2} \quad (18)$$

and the derivative of $\mathbf{W}_1^T(\eta)\mathbf{W}_1(\eta)$ with respect to η produces

$$\frac{\partial}{\partial\eta}[\mathbf{W}^T(\eta)\mathbf{W}(\eta)]$$
$$= \frac{1}{N^2}\sum_{i=0}^{N-1}\frac{\partial}{\partial\eta}\left[1 + \eta\frac{3(N-1)-6i}{N+1}\right]^2$$
$$= \eta\frac{6(N-1)}{N(N+1)}\,. \quad (19)$$

By applying (17)–(19) to (16) and providing the transformations, we derive the optimality coefficient

$$\eta_0 = \frac{\alpha^2 N(N^2-1)}{\alpha^2 N(N^2-1) + 12}\,, \quad (20)$$

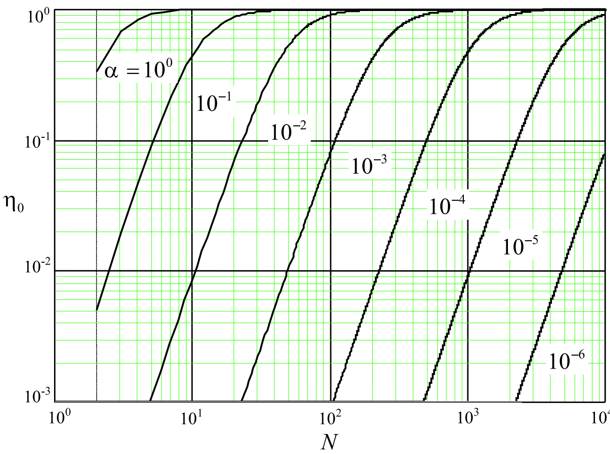

Fig. 2. Dependence of the optimality coefficient η_0 on N and α.

where

$$\alpha^2 = \frac{y_0^2\tau^2}{\sigma_v^2}\,. \quad (21)$$

It is seen that the coefficient η_0 is a function of four variables: the horizon length N, fractional frequency offset y_0, measurement noise variance σ_v^2, and time step τ. If all these values are distinct and so distinct is η, the FIR filter with the kernel (14) will be optimal in the sense of the minimum MSE.

C. An Analysis of an Optimal FIR Filter

Fig. 2 sketches η_0 as a function on N in a wide range of values of α (21). Several important observations can be made using Fig. 2, if we evaluate η_0 for different kinds of clocks. In the below analysis, we imply GPS-based sawtooth measurements with $\tau = 1$s and variance $\sigma_v^2 = \Delta^2/3$ [5], where $\Delta = 50$ns is an upper bound of the sawtooth noise. We also refer to typical offsets y_0 peculated to crystal, rubidium, and cesium clocks.

1) Crystal clocks: For an oven controlled crystal oscillator (OCXO), 10 MHz, we can allow $y_0 = 1.5 \times 10^{-8}$ after 30 days and frequency resolution of 1×10^{-12}. For these values, by (21), we have $\alpha = 0.346$ and $\alpha = 3.46 \times 10^{-5}$ and, by (20), averaging over three hours ($N = 10800$) produces the coefficient η_0 ranging as, respectively,

$$0.999999999920617 > \eta_0 > 0.992124198176391\,.$$

Because $\eta_0 \cong 1$, (14) becomes (13) that makes an unbiased estimate optimal. We thus infer that an unbiased FIR filter may serve as an optimal estimator of the crystal clock TIE.

2) Rubidium clocks: In rubidium clocks, a typical long-term drift is 5×10^{-11}/month. Letting $y_0 = 5 \times 10^{-11}$ and frequency resolution of 1×10^{-13} calculates $\alpha = 1.73 \times 10^{-3}$ and $\alpha = 3.46 \times 10^{-6}$, respectively. Then a one-day average ($N = 86400$) gives, respectively,

$$0.999999993798186 > \eta_0 > 0.998451946589028$$

and we arrive at the same conclusion: the unbiased FIR filter produces virtually an optimal estimate.

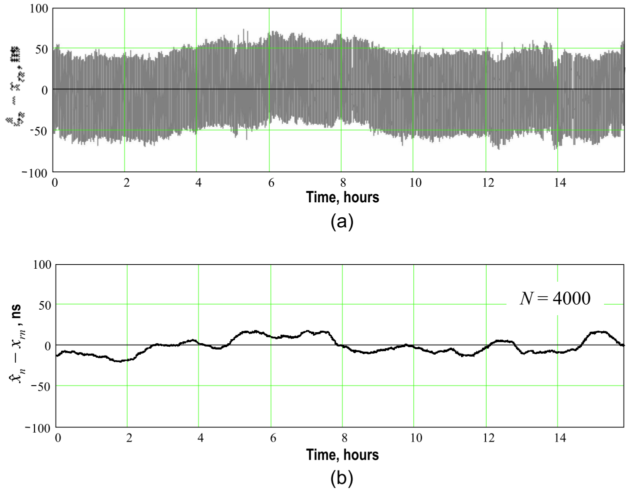

Fig. 3. Error of the TIE measurements and estimation: (a) GPS-based sawtooth measurement error $\xi_n - x_{rn}$ and (b) unbiased FIR estimate error $\hat{x}_n - x_n$.

3) Cesium clocks: For the cesium standard of frequency CsIII, Symmetricom declares 'no aging'. This means that $y_0 = 0$, $\alpha = 0$, and

$$\eta_0 = 0\,.$$

For any horizon N, optimal estimates are thus obtained with simple averaging that is widely used in applications.

From what was observed, we arrive at an important conclusion. There is no reasonable necessity in using optimal filters to estimate the TIE of local clocks. For crystal and rubidium clocks, optimal estimates are provided fairly by unbiased FIR filters and, for cesium clocks, by simple averaging. Note that an optimal FIR (14) requires four variables, whereas an unbiased FIR (13) only one variable.

IV. Applications

To be sure that the above made theoretical inference is correct, we investigated sawtooth measurements of the TIE x_n of a crystal clock imbedded to SR620. We used the Timing SynPaQ III GPS Sensor for GPS-based measurements ξ_n and the Symmetricom frequency standard CsIII for simultaneous reference measurements x_{rn}.

Fig. 3a shows a typical error $\xi_n - x_{rn}$ associated with several hours GPS-based sawtooth measurements. It is seen that the error is mostly caused by sawtooth noise.

Fig. 3b sketches an error $\hat{x}_n - x_{rn}$ of the unbiased FIR estimate obtained by (13) with $N = 4000$. As can be seen, sawtooth noise is well suppressed and the remaining error is caused mostly by the uncertainty of GPS time.

To find an optimal estimate, we use the FIR (14), change gradually the coefficient η from 0 to 1.2, provide FIR estimation, and evaluate the root mean square error (RMSE) in the estimate. Fig. 4 shows what happens with the RMSE, by different η. We see that the minimum RMSE of 9.7 ns corresponds to an unbiased estimate making it optimal. Apart the point $\eta = 1$, the RMSE grows dramatically owing to increase in the estimate bias. Inherently, the RMSE reaches a maximum at $\eta = 0$ caused by the large (50%) bias in simple averaging.

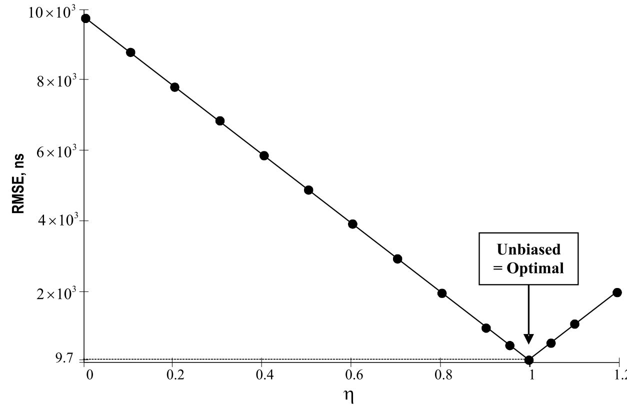

Fig. 4. RMSEs of FIR estimates for different optimality coefficients η.

V. Conclusions

In this paper, we derived an optimal FIR filter for linear TIE models and compared optimal estimates to those obtained by the unbiased FIR filter. We showed theoretically and verified experimentally that there is no reasonable necessity in using optimal filters for estimation of the local clock TIE. For crystal and rubidium clocks, optimal estimates are provided fairly by unbiased FIR filters and, for cesium clocks, by simple averaging. Note that an optimal FIR (14) requires four variables, whereas an unbiased FIR (13) only one variable.

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